Monetary Misperceptions: Optimal Monetary Policy under Incomplete Information

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Abstract

Inflation targeting is strictly suboptimal when economic actors have incomplete information about the state of the economy. Nominal income targeting is approximately optimal, and exactly optimal under certain parameterizations. We derive this result in a "Lucas islands" monetary misperceptions model built from, unlike prior work, explicit microfoundations. Agents have knowledge of *local* productivity and money supply conditions, but must perform a signal extraction problem each period to estimate the *aggregate* productivity shock and the *aggregate* money supply shock. Without full information, agents cannot perfectly distinguish between relative price shocks and aggregate shocks, causing monetary policy to affect the real economy. Since the model is built from agents optimizing from first principles, we are able to take a second-order welfare approximation and ask what monetary policy rule is optimal. In contrast to sticky price or sticky information models, inflation and price level targeting are always suboptimal, as price level variation provides useful information to agents. Under log utility, nominal income targeting is optimal.

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1 Introduction

What should a central bank do? Most central banks, at least in developed nations, target a low and stable rate of inflation. This conception of inflation targeting as the optimal monetary policy target can be justified by the workhorse model of modern monetary economics, the sticky price general equilibrium ("New Keynesian") model. In this model, optimal monetary policy is to target a zero rate of inflation, which happens to simultaneously prevent both recessions and unsustainable booms. Similarly, the popular "sticky information" model prescribes price level targeting as optimal monetary policy, a related policy. The optimality of inflation and price level targeting in these models is a result of the specific frictions in these model (sticky prices and sticky information, respectively).

This paper shows that the optimality of inflation targeting is not robust to the choice of friction. In our model, agents have incomplete information about the state of the economy. In particular, agents are able to observe local money supply, money demand, and productivity, but must estimate the level of the aggregate money supply, aggregate money demand, and aggregate productivity. Under these conditions, inflation or price level targeting is actively harmful: by being unnecessarily active in stabilizing the price level, central banks mute valuable information contained in the inflation which results from aggregate productivity shocks.

Optimal policy is instead approximately nominal income targeting, and under certain parameterizations is precisely nominal income targeting. More generally, the central bank should allow the price level to fall in response to technological innovations, and conversely allow the price level to rise in response to negative productivity shocks. The aggregate price level thus acts as a coordination mechanism, analogous to the way that relative prices convey useful information to agents about the relative scarcity of different goods. For example, when productivity falls and aggregate output is more scarce, the aggregate price level signals this by rising.

We establish this result in a "Lucas Islands" model. One contribution of this paper is to revive the Lucas islands model and to fully ground it in a modern optimizing framework. Whereas previous work (Lucas (1972); Lucas (1973); Barro (1976); McCallum (1984)) simply postulated some of the basic economic relationships of the model, we start from heterogeneous agents optimizing to maximize their expected utilities, with money demand resulting from a cash-in-advance (CIA) constraint.

Lucas described a world composed of many isolated islands with each island producing a different good. Agents on any given island are aware of economic conditions on their specific island, but are unaware of aggregate economic conditions.

As a result of the isolation of islands, if the central bank boosts the aggregate money supply across all islands, any individual agent may misperceive the resulting increase in nominal spending as an increase in real demand for their island's *specific* good, rather than merely a nominal change, causing them to increase production. In this way, nominal variables can affect real variables.

Lucas' model was meant to capture the very realistic problem of information frictions. Consider the owner of an isolated bakery. Suppose one day, all of the customers seen by the baker spend twice as much money as the customers from the day before. The baker has two options. She can interpret this increased demand as customers having come to appreciate the superior quality of *her* baked goods, and increase her production to match the new demand. Alternatively, she could interpret this increased spending as evidence that there is simply more money in the economy *as a whole*, and that she should merely increase her prices proportionally to account for inflation. Economic agents confounding these two effects is the source of economic booms and busts, according to this logic. On the other hand, if the central bank had announced that the money supply had been doubled yesterday, the baker could have fully anticipated the new spending and raised prices appropriately.

Using this framework, we solve for "full information" output, that is, the natural rate of output which would occur in the absence of any information frictions. Because the model is fully microfounded from optimizing agents, we are then able to take a second degree approximation of the agents' utility functions to get a measure of welfare in the style of Woodford (2002) (see also Ball et al. (2005); Ravenna and Walsh (2003)), which turns out to be a function of the squared difference between actual output and full information output. This allows us to ask what monetary policy rule is welfare-optimal, as outlined above.

In the reasonable case that aggregate technology shocks have a unit root – that is, if an invention is made today, it is not forgotten immediately tomorrow or slowly forgotten over time – then optimal policy approaches nominal income (NGDP) targeting as information approaches completeness. Nominal income targeting is also exactly optimal if agents have log preferences over consumption.

In the more general case where technology shocks follow a first-order Markov process, optimal monetary policy does not merely target a constant inflation rate or price level, but instead is more flexible. In other words, the strict inflation targeting pursued by some real-world central banks is rejected.

This paper proceeds as follows. The second brief section runs through the set-up of a standard cash-in-advance model without capital or credit and linearizes it around the steady state, and the third section incorporates the monetary misperceptions friction into this model. The fourth section defines full-information output, and the fifth section derives a second-order welfare approximation using this definition. The sixth section discusses the recursive laws of motion, and the seventh section uses this system and the welfare approximation to solve for optimal monetary policy. An eighth section extends the model by allowing for variation in velocity. A discussion of optimal policy and conclusion follow.

2 Model core: A CIA model

The core of the model is a simple cash-in-advance model of Lucas and Stokey (1987) without capital. The economy is a purely cash economy; there is no credit. Because these results are standard, available in any textbook (e.g. Walsh (2010)), and not a contribution of this paper, we present most results without derivation. We begin by discussing a representative household and later add the heterogeneity.

The representative household maximizes utility, which is increasing in consumption of the single consumption good c_t and decreasing in labor n_t.

$$\max_{\{c_{t+i}, n_{t+i}, m_{t+i}, b_{t+i}\}_{i=0}^{\infty}} \quad E_t\left[\sum_{i=0}^{\infty} \frac{1}{1-\sigma} c_{t+i}^{1-\sigma} - \frac{\chi}{1-\eta} n_{t+i}^{1+\eta}\right] \tag{1}$$

The household faces the CIA constraint $p_t c_t \leq M_{t-1} + T_t$, where p_t is the nominal price of the consumption good, M_{t-1} is the supply of money carried over from the previous period, and T_t is the nominal money supply transfer from the central bank. Defining real variables $m_{t-1} = \frac{M_{t-1}}{p_{t-1}}, \tau_t = \frac{T_t}{p_t}$ and inflation $\pi_t = \frac{p_t - p_{t-1}}{p_{t-1}}$, then this constraint can be written in real terms as

$$c_t \leq \frac{m_{t-1}}{1+\pi_t} + \tau_t \tag{2}$$

The household allocates wealth today among consumption, savings in the form of bonds B_t, and money M_t for next period. Wealth today consists of income y_t, interest income from the interest rate i_{t-1} on last period's savings, money carried over from last period, and the central bank's money supply transfer today. In real terms, where $b_t = \frac{B_t}{p_t}$, the budget constraint can be written as

$$c_t + b_t + m_t \leq y_t + \left(\frac{1+i_{t-1}}{1+\pi_t}\right) b_{t-1} + \frac{m_{t-1}}{1+\pi_t} + \tau_t \tag{3}$$

Finally, the production function is linear in labor, with stochastic technology shock term a_t.

$$y_t = a_t n_t \tag{4}$$

The household maximizes (1) subject to (2)-(4). Denoting the Lagrangian multipliers on the budget constraint and the CIA constraint λ_t and μ_t respectively, it is straightforward to show using a Bellman equation that the equilibrium conditions are as follows:

$$c_t^{-\sigma} = \lambda_t + \mu_t \tag{5}$$

$$\chi n_t^{\eta} = \lambda_t \frac{y_t}{n_t} \tag{6}$$

$$\lambda_t = \beta E_t \left[\frac{\lambda_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}} \right] \tag{7}$$

$$\lambda_t = \beta E_t \left[\lambda_{t+1} \frac{1 + i_t}{1 + \pi_{t+1}} \right] \tag{8}$$

The first equation shows that the household equates the marginal utility of consumption to the marginal utility of wealth (λ_t) plus a wedge induced by the CIA constraint (μ_t). The second equation is the marginal rate of substitution condition. The third equation implicitly defines money demand. The fourth equation is the consumption Euler equation.

Market clearing implies $c_t = y_t$ and $T_t = M_t - M_{t-1}$.

Appendix A log linearizes this model around the steady state. Define the percentage deviation of variable x from its steady state value x^{ss} as $\hat{x}_t = \frac{x_t}{x^{ss}} - 1$. We apply this to all variables excepting interest rates and inflation, which are already percentages, where we instead define $\hat{i}_t = i_t - i^{ss}, \hat{\pi}_t = \pi_t - \pi^{ss}$. Appendix A derives that:

$$\hat{y}_t = \hat{n}_t + \hat{a}_t \tag{9}$$

$$\hat{y}_t = \hat{c}_t = \hat{M}_t - \hat{p}_t \tag{10}$$

$$\hat{\lambda}_t = E_t\left[-\sigma \hat{y}_{t+1} - \hat{\pi}_{t+1}\right] \tag{11}$$

$$(1+\eta)\hat{n}_t = \hat{y}_t + \hat{\lambda}_t \tag{12}$$

$$\hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \hat{i}_t - E_t\hat{\pi}_{t+1} \tag{13}$$

$$E_t\hat{\mu}_{t+1} = E_t\hat{\lambda}_{t+1} + \frac{1-\Phi}{\Phi}E_t\hat{i}_t \tag{14}$$

(9) is the linearized production function. (10) follows from market clearing and the result that in an equilibrium with positive nominal interest rates, the CIA constraint binds. The marginal utility of wealth (11) is implied by (5), (7), and market clearing. (12) is the linearized MRS condition, and (13) is the linearized Euler equation. (14) is the result of combining (7) and (8), where $\Phi \equiv 1 - \frac{\beta}{1+\pi^{ss}} = \frac{i^{ss}}{1+i^{ss}}$ measures steady-state deviation from the Friedman rule ($\Phi = 0$ implies the Friedman rule is implemented).

3 Incorporating monetary misperceptions

Thus far, the model presented is entirely classical. We now add the imperfect information friction that will generate monetary misperceptions.

There is a population of agents who each live on separate islands, and only have knowledge about economic conditions on their local island. Agent i lives in a cash-in-advance economy on her respective island i and produces differentiated output y_t^i. To make money demand stochastic, agents are randomly reallocated among islands after each period. Agents are equally likely to be distributed to any particular island.[1] *As a result, when agents optimize, they care about local economic conditions for date t variables, but care about* <u>*aggregate*</u> *economic conditions for date $t+1$ variables.*

[1] In the original formulation of Lucas, agents had two-period lives, and young agents were distributed randomly to each location. My infinitely-lived framework makes welfare analysis more tractable. See also (Walsh, 2010).

We denote local variables with a superscript i, and aggregate variables without a superscript. Our linear system of (9)-(14), which represented the classical CIA model, is thus transformed into the Lucas islands model as follows:

$$\hat{y}_t^i = \hat{n}_t^i + \hat{a}_t + \hat{\aleph}_t^i \tag{15}$$

$$\hat{y}_t^i = \hat{M}_t^i - \hat{p}_t^i \tag{16}$$

$$\hat{\lambda}_t^i = E_t\left[-\sigma \hat{y}_{t+1} - \hat{\pi}_{t+1}\right] \tag{17}$$

$$(1+\eta)\hat{n}_t^i = \hat{y}_t^i + \hat{\lambda}_t^i \tag{18}$$

$$\hat{\lambda}_t^i = E_t\hat{\lambda}_{t+1} + \hat{i}_t^i - E_t\hat{\pi}_{t+1} \tag{19}$$

$$E_t\hat{\mu}_{t+1} = E_t\hat{\lambda}_{t+1} + \frac{1-\Phi}{\Phi}\hat{i}_t^i \tag{20}$$

3.1 Misperceptions on productivity shocks

Note that the local production function (15) is now $y_t^i = a_t \aleph_t^i n_t^i$. That is, each good in the economy experiences an aggregate productivity shock a_t as well as an idiosyncratic productivity shock $\aleph_t^i$. We impose that idiosyncratic productivity shocks are serially uncorrelated white noise with variance $\sigma_\aleph^2$ and which average to zero across islands. Further, aggregate productivity shocks follow a first-order Markov chain process: where $\varepsilon_t^a \sim N(0, \sigma_a^2)$ is white noise,

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_t^a \tag{21}$$

Agents have incomplete information, and cannot distinguish between local productivity shocks and aggregate productivity shocks. That is, since an agent on island i can observe local output $\hat{y}_t^i$, they can infer $\left(\hat{a}_t + \hat{\aleph}_t^i\right)$, but not the individual components separately. This will be very important.

We suppose agents have rational expectations, with the underlying parameters of the economy known.[2] Agents then estimate the aggregate productivity shock using the linear least-squares estimator:

$$E_t^i \hat{a}_t = \omega \left(\hat{a}_t + \hat{\aleph}_t^i \right) \tag{22}$$

where $\omega = \sigma_a^2/(\sigma_a^2 + \sigma_\aleph^2)$. Note that $0 \leq \omega \leq 1$, and if aggregate productivity shocks are large compared to local productivity shocks, agents will attribute more of the combined productivity shock to the aggregate $\hat{a}_t$. Vice versa, if idiosyncratic productivity shocks are large, then ω will be close to zero.

3.2 Misperceptions on money supply shocks

We have yet to specify a monetary policy rule for how the central bank will set the nominal money supply. We suppose a rule of the following form for the aggregate money supply:

$$\hat{M}_t = \rho_m \hat{M}_{t-1} + v_t + u_t + \phi \hat{a}_t \tag{23}$$

Here, ϕ is the response of the central bank to productivity shocks. v_t and u_t are both serially uncorrelated white noise money supply shocks. What distinguishes them is that v_t is public information whereas u_t is not. In other other words, v_t is known to agents, but agents will have to estimate u_t just as they have to estimate a_t. This estimation is detailed further momentarily.

The nominal money supply on island i follows:

$$\hat{M}_t^i = \rho_m \hat{M}_{t-1} + v_t + u_t + u_t^i + \phi \hat{a}_t + \phi^i \hat{\aleph}_t^i \tag{24}$$

Here, ϕ^i is the response of the central bank to idiosyncratic local productivity shocks. u_t^i is an island-specific money supply shock. Like u_t, it is not public information. Like $\hat{\aleph}_t^i$, we suppose that it is white noise which averages to zero.

[2] Incorporating how agents learn these parameters over time could be a very fruitful extension of this framework.

As a result, agents on island i must perform a signal extraction problem when estimating the value of u_t, analogous to the same problem faced with estimating $\hat{a}_t$. Denoting the variances of u_t and u_t^i as σ_u^2 and σ_i^2, respectively, then:

$$E_t^i u_t = \kappa(u_t + u_t^i) \tag{25}$$

where $\kappa = \sigma_u^2/(\sigma_u^2 + \sigma_i^2)$. The same interpretation applies as with the least-squares estimator for the aggregate productivity shock.

3.3 Summing up

The system representing the Lucas islands model consists of equations (15)-(25).

In a subsequent section, we will derive expressions for the price level and output as a function of the state variables $(\hat{M}_{t-1}, \hat{a}_t)$ and exogenous shocks $(v_t, u_t, u_t^i, \hat{\aleph}_t^i)$. We first discuss what the behavior of output would be in the absence of the information frictions, since this will make the expressions for the price level and output exceedingly more intuitive.

4 Unconstrained full-information output

Define "unconstrained, full-information" output as the level of output which would prevail in the absence of both the CIA constraint and the Lucas island information friction. That is, the unconstrained full-information output level is the level of output which would prevail in the most basic real business cycle model. Denote the aggregate level of unconstrained, full-information output as $\hat{y}_t^f$ and island i's unconstrained, full-information output level as $\hat{y}_t^{i,f}$.

Appendix B solves for this level of output by equating the marginal rate of substitution (in the absence of the CIA constraint) to the marginal productivity of labor. The result is:

$$\hat{y}_t^{i,f} = \frac{1+\eta}{\sigma+\eta}\left[\hat{a}_t + \hat{\aleph}_t^i\right] \tag{26}$$

$$\hat{y}_t^{f} = \frac{1+\eta}{\sigma+\eta}\hat{a}_t \tag{27}$$

That is, the unconstrained, full-information level of output at time t is directly proportional to the productivity shock of time t.

5 The welfare approximation

Following Woodford (2002), we take a second order approximation of the utility function to get a tractable representation of welfare of agents in the economy. We can then ask what type of monetary policy rule maximizes this welfare approximation.

Define the deviation of utility from steady state as
$U_t^i - U^{i,ss} \equiv U(c_t^i, n_t^i) - U(c^{i,ss}, n^{i,ss}) = \frac{1}{1-\sigma}\left({c_t^i}^{1-\sigma} - {c_t^{i,ss}}^{1-\sigma}\right) - \frac{\chi}{1+\eta}\left({n_t^i}^{1+\eta} - {n_t^{i,ss}}^{1+\eta}\right)$
Appendix C shows, after much tedious algebra, that this has the second-order approximation of

$$\xi\left[U_t^i - U^{i,ss}\right] = -\frac{1}{2}\left[\hat{y}_t^i - \hat{y}_t^{i,f}\right]^2 + t.i.p. \tag{28}$$

where $t.i.p.$ is "terms independent of monetary policy", that is, terms which cannot be affected by monetary policy. ξ is a constant.

Period utility thus is solely a function of the deviation of output from its unconstrained, full-information level. Any deviation – up or down – is welfare-reducing.

It is worthwhile to note what does *not* show up in this approximation. Note that the variance of the price level does not appear. That is, the agent has no instrumental reason to care about price variability (i.e. inflation), except inasmuch it causes output to deviate from its "natural" unconstrained level.

This is contrast to many similar papers which also take a second-order approximation of the utility function and ask what monetary policy rule is welfare-maximizing (e.g. Woodford (2002); Ravenna and Walsh (2003); Ball et al. (2005)). This is because these models are built on top of a monopolistic competition foundation, where agents consume a composite consumption good, typically of Dixit-Stiglitz form $c_t = \left[\int_0^1 c_{it}^{1/\theta} di\right]^\theta$. Agents have a "taste for variety", as different goods c_{it} and c_{jt} are complements for one another. Cross-sectional price dispersion distorts the relative prices of these differentiated goods, causing the agent to consume amounts of the individual goods which differ from the optimum flex-price quantity. Because of diminishing marginal utility, this lowers welfare. Thus, the second-order welfare approximation includes a term for the cross-sectional variance of inflation.

This does not appear here. Each island's output y_t^i could itself be a composite good, $y_t^i = \left[\int_0^1 y_t^i(j)^{1/\theta} dj\right]^\theta$, but without any friction to distort the composition of this composite (e.g. the Calvo sticky price friction so popular in the models cited above), this complication would have precisely zero affect on welfare. A more promising approach may be the incorporation of the state-dependent Ss pricing model of Gertler and Leahy (2008). This model features separated islands with each island containing a continuum of monopolistically competitive firms, with state-dependent sticky prices resulting from idiosyncratic productivity shocks interacting with fixed adjustment costs.

6 The recursive laws of motion

We now return to the equilibrium conditions of the model (15)-(25). With full-information output and the welfare approximation established to provide intuition for what we are about to see, we are now able to examine the recursive laws of motion for the price level and output.

Appendix D uses the method of undetermined coefficients to prove that the price level can be written as a function of state variables $\hat{M}_{t-1}, \hat{a}_t$ and exogenous shocks $v_t, u_t, u_t^i, \hat{\aleph}_t^i$. It shows that

$$\hat{p}_t^i = \rho_m \hat{M}_{t-1} + v_t + \frac{\eta + \kappa}{\eta + 1} \left(u_t + u_t^i\right) + a_5 \hat{a}_t + a_6 \hat{\aleph}_t^i \tag{29}$$

and thus

$$\hat{y}_t^i = \hat{M}_t^i - \hat{p}_t^i = \left[\frac{1-\kappa}{\eta+1}\right] (u_t + u_t^i) + (\phi - a_5)\hat{a}_t + (\phi^i - a_6)\hat{\aleph}_t^i \tag{30}$$

where

$a_5 = -\frac{\eta+1}{\eta+1+(\sigma-1)\omega\rho_a} + \phi\frac{\eta+\omega+(\sigma-1)\omega\rho_a}{\eta+1+(\sigma-1)\omega\rho_a}$

$a_6 = -1 + \frac{\eta}{\eta+1}\phi^i + \phi\frac{\omega+(\sigma-1)\omega\rho_a}{\eta+1} - \frac{\sigma-1}{\eta+1}\omega\rho_a a_5$

The coefficients in (29) and (30) have intuitive explanations. Note first that the price level, (29), adjusts one-for-one with predictable or announced money supply changes: the coefficients on $\rho_m \hat{M}_{t-1}$ and v_t are both unity. As a result, anticipated money supply shocks have no effect on output in (30).

In contrast, money supply shocks do affect output if they are unannounced and there is incomplete information ($\kappa < 1$). That is, unannounced money supply shocks, $u_t + u_t^i$, affect output, and with a larger effect the smaller that κ is. Agents are unable to completely distinguish between purely nominal inflationary shocks and relative price movements. To walk through an example, suppose an agent on island i sees the price of local consumption good c_t^i increase. They will not be able to perfectly accurately tell if this price increase indicates higher relative demand for their output, or if instead it is merely a nominal price increase of no real significance.

Interpreting the relationship between the price level and productivity shocks is slightly more complicated. It is useful to discuss an edge case to build intuition.

Suppose that aggregate productivity follows a random walk ($\rho_a = 1$), and that agents have perfect information ($\omega = 1$). In this case, $a_5 = -\frac{1+\eta}{\sigma+\eta} + \phi$. As a_5 is the coefficient on aggregate productivity in the equation for the price level, this means that the price level adjusts negative one-for-one with potential aggregate output (recall $\hat{y}_t^f = \frac{1+\eta}{\sigma+\eta}\hat{a}_t$) distorted by however much the monetary authority responds to supply shocks, ϕ.

From (29) and (30), it follows immediately that the *aggregate* price level and *aggregate* output are

$$\hat{p}_t = \rho_m \hat{M}_{t-1} + v_t + \frac{\eta+\kappa}{\eta+1}u_t + a_5\hat{a}_t \tag{31}$$

$$\hat{y}_t = \hat{M}_t - \hat{p}_t = \frac{1-\kappa}{\eta+1}u_t + (\phi - a_5)\hat{a}_t \tag{32}$$

We turn next to the question of, what monetary policy rule should the central bank follow in order to maximize welfare? Put in more mathematical terms: what are the optimal coefficients ϕ and ϕ^i, and what are the optimal time-paths for v_t, u_t, and u_t^i?

7 Optimal monetary policy

7.1 Optimal monetary policy shocks

Trivially, the optimal monetary policy rule will not engage in unanticipated shocks to the money supply: i.e., the central bank should set $u_t = u_t^i = 0$ for all t. Unanticipated money shocks only create noise and are welfare-reducing.

On the other hand, pre-announced monetary policy changes v_t do not affect local or aggregate output no matter what. As a result, *announced* monetary policy changes can be as noisy as the central banker wants, without impacting welfare.

However, one can envision that this result would not hold were only the smallest adjustment costs added to this framework. In other words, although this model implies that optimal policy allows for any monetary policy rule as long as it is pre-announced (imagine, as a clearly absurd example, a k-percent rule for the money supply where k is determined by the month of the year) there are good reasons *outside of this model* to think that that unnecessary pre-announced monetary changes should be kept to a minimum.

7.2 Optimal response to supply shocks

Optimal policy becomes interesting when the central bank must answer the question of how to respond to supply shocks: what are the optimal productivity response coefficients ϕ and ϕ^i?

The central bank seeks to maximize aggregate welfare, $\int_i \left[U_t^i - U^{i,ss}\right] di$. We discussed previously the result that individual welfare can be approximated as $\xi\left[U_t^i - U^{i,ss}\right] = -\frac{1}{2}\left[\hat{y}_t^i - \hat{y}_t^{i,f}\right]^2 + t.i.p.$, where $t.i.p.$ is "terms independent of monetary policy", i.e. terms not affected by the central bank. We can use this result to get an expression for aggregate welfare.

Differencing the expressions for $\hat{y}_t^i$ and $\hat{y}_t$ from (30) and (32), we find that $\hat{y}_t^i = \hat{y}_t + \frac{1-\kappa}{1+\eta}u_t^i + (\phi^i - a_6)\hat{\aleph}_t^i$. Similarly differencing the expressions for $\hat{y}_t^{i,f}$ and $\hat{y}_t^f$ from (26) and (27), we find that $\hat{y}_t^{i,f} = \hat{y}_t^f + \frac{1+\eta}{\sigma+\eta}\hat{\aleph}_t^i$. Thus

$$\hat{y}_t^i - \hat{y}_t^{i,f} = \hat{y}_t - \hat{y}_t^f + \frac{1-\kappa}{1+\eta}u_t^i + (\phi^i - a_6)\hat{\aleph}_t^i - \frac{1+\eta}{\sigma+\eta}\hat{\aleph}_t^i$$

Then, turning off money supply shocks since we know these are suboptimal, $\left[\hat{y}_t^i - \hat{y}_t^{i,f}\right]^2 = \left[\hat{y}_t - \hat{y}_t^f\right]^2 + 2\left(\hat{y}_t - \hat{y}_t^f\right)\left(\phi^i - a_6 - \frac{1+\eta}{\sigma+\eta}\right)\hat{\aleph}_t^i + \left(\phi^i - a_6 - \frac{1+\eta}{\sigma+\eta}\right)^2\hat{\aleph}_t^{i2}$ Integrating over i, we find an expression for aggregate welfare in terms of aggregate output, aggregate full-information unconstrained output, and terms independent of policy:

$$
\begin{aligned}
\xi \int_i \left[U_t^i - U^{i,ss}\right] di &= \int_i \left\{ -\frac{1}{2} \left[\hat{y}_t^i - \hat{y}_t^{i,f}\right]^2 + t.i.p \right\} \\
&= \int_i \left[\hat{y}_t - \hat{y}_t^f\right]^2 di + 2\left(\hat{y}_t - \hat{y}_t^f\right)\left(\phi^i - a_6 - \frac{1+\eta}{\sigma+\eta}\right)\int_i \hat{\aleph}_t^i di \\
&\quad + \left(\phi^i - a_6 - \frac{1+\eta}{\sigma+\eta}\right)^2 \int_i \hat{\aleph}_t^{i^2} di \\
&= \int_i \left[\hat{y}_t - \hat{y}_t^f\right]^2 di + \left(\phi^i - a_6 - \frac{1+\eta}{\sigma+\eta}\right)^2 \int_i \hat{\aleph}_t^{i^2} di
\end{aligned} \tag{33}
$$

Where the last line follows from the fact that idiosyncratic productivity shocks average to zero across islands.

It is possible for the central bank to replicate the unconstrained, full-information equilibrium by its choice of ϕ and ϕ^i. This will result in aggregate welfare's deviation from steady state, as specified in the above equation, always being exactly zero.

First, the first term in (33), the deviation of aggregate output from its natural level, can be zeroed by an appropriate choice of ϕ. From (32) and (27), the optimal central bank response to productivity shocks ϕ^* will satisfy

$$
\begin{aligned}
\phi^* - a_5 - \frac{1+\eta}{\sigma+\eta} &= 0 \\
\Longrightarrow \phi^* &= \frac{1+\eta}{\sigma+\eta}\frac{\eta+1+(\sigma-1)\omega\rho_a}{1-\omega} - \frac{1+\eta}{1-\omega}
\end{aligned} \tag{34}
$$

Second, given ϕ, then ϕ^i can be chosen to zero the second term of (33):

$$
\begin{aligned}
\phi^{i^*} &= a_6 + \frac{1+\eta}{\sigma+\eta} \\
\Longrightarrow \phi^{i^*} &= (\eta+1)\left\{ \frac{-\eta-1+(\omega-1)\rho_a(\sigma-1)}{\eta+1+(\sigma-1)\rho_a} \right. \\
&\quad \left. + \frac{1+\eta}{\sigma+\eta}\left[\frac{\omega+(\sigma-1)\omega\rho_a}{\eta+1} - \frac{\sigma-1}{\eta+1}\omega\rho_a \frac{\eta+\omega+(\sigma-1)\omega\rho_a}{\eta+1+(\sigma-1)\omega\rho_a}\right]\phi^* \right\}
\end{aligned}
$$

Discussion is postponed to section 9. We first introduce velocity shocks.

8 Introducing velocity shocks

Velocity shocks can be introduced by allowing the cash-in-advance constraint to vary over time in its bindingness. Instead of equation (16), $\hat{y}_t^i = \hat{M}_t^i - \hat{p}_t^i$, suppose that there is exogenous velocity $x_t + x_t^i$:

$$\hat{y}_t^i = \hat{M}_t^i - \hat{p}_t^i + \hat{x}_t + \hat{x}_t^i \tag{35}$$

where, analogous to productivity shocks,

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_t^x \tag{36}$$

and $\varepsilon_t^x \sim N(0, \sigma_x^2)$ is white noise. Idiosyncratic velocity shocks $\hat{x}_t^i$ are uncorrelated with other shocks, have variance $\sigma_{x^i}^2$, and average to zero across islands.

As with money supply and productivity, agents are only able to observe the sum $(\hat{x}_t + \hat{x}_t^i)$, but not the individual components. They rationally estimate aggregate velocity as

$$E_t^i \hat{x}_t = \gamma \left(\hat{x}_t + \hat{x}_t^i\right) \tag{37}$$

where $\gamma = \sigma_x^2 / \left(\sigma_x^2 + \sigma_{x^i}^2\right)$.

The money supply rule now allows the central bank to respond to velocity shocks:

$$\hat{M}_t = \rho_m \hat{M}_{t-1} + v_t + u_t + \phi \hat{a}_t + \psi \hat{x}_t \tag{38}$$

$$\hat{M}_t^i = \rho_m \hat{M}_{t-1} + v_t + u_t + u_t^i + \phi \hat{a}_t + \phi^i \hat{\aleph}_t^i + \psi \hat{x}_t + \psi^i \hat{x}_t^i \tag{39}$$

The refinement of variable velocity does not affect full-information unconstrained output or the welfare approximation, but the recursive laws of motion for prices and output are altered. The expressions for price level and output, (31) and (32), gain an additional term to account for the effect of velocity on prices and output, becoming:

$$\hat{p}_t = \rho_m \hat{M}_{t-1} + v_t + \frac{\eta + \kappa}{\eta + 1} u_t + a_5 \hat{a}_t + a_7 \hat{x}_t \tag{40}$$

$$\hat{y}_t = \hat{M}_t - \hat{p}_t = \frac{1 - \kappa}{\eta + 1} u_t + (\phi - a_5)\hat{a}_t + (1 + \psi - a_7)\hat{x}_t \tag{41}$$

where a_5 is as defined before, and:

$a_7 = \frac{\eta+\gamma\sigma\rho_x}{\eta+\gamma\sigma\rho_x-\gamma\rho_x+1} + \psi\frac{\eta+\gamma\sigma\rho_x-\gamma\rho_x+\gamma}{\eta+\gamma\sigma\rho_x-\gamma\rho_x+1}$

The addition of variable velocity does not affect optimal policy with regards to monetary shocks or the response to supply shocks (i.e., u_t^*, v_t^*, ϕ^* are unaffected). As for optimal monetary policy response to velocity shocks, ψ^*, the central bank again seeks to ensure that realized output follows the natural level. This occurs if the central bank responds to velocity shocks so that velocity shocks have precisely zero impact on output. This can be achieved by setting:

$$\psi^* = -\frac{\gamma\rho_x - 1}{\gamma - 1} \tag{42}$$

9 Discussion of optimal policy

The solution for optimal monetary policy when agents have incomplete information about the state of the economy is the main contribution of this paper. Below, we discuss the relation between optimal policy in our environment and two popular monetary policy targets: inflation targeting and nominal income (NGDP) targeting. We also discuss time consistency and implementability issues facing monetary policymakers.

9.1 Optimal policy's relation to nominal income targeting

Prior authors have asserted that in models of monetary misperceptions, nominal income targeting is optimal policy (cf. Selgin (1997)). According to the logic above, this is *almost* the case.

Nominal income targeting would be optimal if $\phi^* = 0$ and $\psi^* = -1$. That is, nominal income targeting is optimal if the central bank should not vary the money stock in response to supply shocks, but should offset changes in money demand one-for-one.

In the case where shocks follow a unit root (i.e. $\rho_a = \rho_x = 1$), this is nearly the case. Indeed, in this case, we do have $\psi^* = -1$. However, optimal policy would not have the central bank completely ignore supply shocks: $\phi^* \neq 0$. However, optimal policy approaches nominal income targeting as information approaches completeness.

In the more general case where shocks follow a Markov process, policy is likely to be quite close quantitatively to nominal income targeting. Further, if the central bank does not have perfect information about the structural parameters of the economy, then nominal income targeting may be superior to fine-tuning of the central bank's reaction function (see below for further discussion).

Finally, nominal income targeting is always optimal if agents have log preferences over consumption: in the limit as σ approaches unity (i.e., log utility), then $\phi^* = 0$.

Why is nominal income targeting not precisely optimal more generally?

The intuition for the optimality of nominal income targeting comes from the following idea: in the face of a technology shock, the central bank should not alter the money supply (i.e. ϕ should be 0), since the central bank cannot affect the supply-side of the economy but only the demand-side. Otherwise put, one might expect that if the central bank does nothing in response to a technology shock, then output would immediately jump to its new natural level.

However, output – at least in the framework outlined above – cannot jump precisely to its new natural level. This is because agents can only *estimate* exactly what the productivity shock today is, so they can only *estimate* what the new natural rate of output is, and jump to that. The central bank can correct this by its choice of ϕ and ensure that, following a technology shock, output jumps immediately to its new natural level.

9.2 Optimal policy and inflation targeting

It is worthwhile to highlight the fact that optimal policy differs from the strict inflation targeting that is often advocated, e.g. from the most basic sticky price model.[3] In a sticky price model, in the case of the negative supply shock mentioned above, the optimal policy rule has the central bank *raise* interest rates to *lower* inflation back to zero. In our model, that would be highly suboptimal and would induce monetary misperceptions. By keeping the price level constant, agents would not be able to perceive the true change in productivity, and output would be reduced below its full-information level and welfare would be reduced.

This difference has important real-world consequences. For example, the European Central Bank (ECB) strictly targets inflation, more so than other central banks. In 2011, the ECB chose to raise its policy rate in the face of rising prices due to a negative supply shock – rising oil prices. This is logical under strict inflation targeting. However, monetary misperceptions theory would have advocated instead that the ECB allow prices to rise temporarily, so as to signal the negative supply shock, as described above. Indeed, following the ECB's rate hike, the eurozone was plunged into a double-dip recession from which it still struggles to recover.

[3] i.e., the New Keynesian model with a single composite good and the Calvo staggered price friction. As Selgin (1997) discusses, sticky prices do not *necessarily* imply the optimality of strict zero inflation targeting if (1) the price stickiness is the result of menu costs, (2) there are heterogeneous goods, and (3) there are both aggregate and idiosyncratic shocks.

9.3 Time consistency and implementability

The optimal policy described above is time consistent. Monetary misperceptions in this model only last one period, so the dynamics are limited. Additionally, the welfare approximation is a negative function of the deviation of output from its full-information level, *squared*. This means that the central bank has no incentive to push output above its full-information level. This is in contrast to a more naive central bank objective function not derived from first principles which would merely seek to maximize output.

A more pertinent critique of the optimal policy described above is implementability. Except under log utility or in the limiting case, optimal policy requires the central bank respond, to some extent, to productivity shocks ($\phi^* \neq 0$). As a result, the central bank must be able to measure the exact size of the technology shock – that is, it must know the true potential output of the economy. For standard Hayekian reasons, this is not feasible (see further discussion in Halperin 2015). It must be noted, however, that any model which prescribes that the central bank follow a Taylor Rule that responds to potential output will fall victim to this same critique.[4]

10 Conclusion

In the metaphor of Selgin (1997), consider listening to a symphony on the radio. Randomly turning the volume knob up and down merely detracts from the musical performance (random variation in the price level is not useful). But, the changing volume of the orchestra players themselves, from quieter to louder and back down again, is an integral part of the performance (the price level *should* adjust with

[4]See Beckworth and Hendrickson (2016) for a calibrated sticky price model where the central bank must estimate potential output with error.

natural variations in output). The changing volume of the orchestra should not be smoothed out to maintain a constant volume (constant inflation is not optimal).

This paper makes this argument rigorously in a full-scale DSGE framework. We build a model from first principles where agents have only information about local economic conditions – productivity, money supply, money demand – and limited information about these variables in aggregate. We show that the welfare of these agents is maximized when output is at the level which would obtain in the absence of said information frictions. This can be achieved through appropriate monetary policy, specifically, a policy which allows for some aggregate price level variation in response to technology shocks.

There are several natural next steps. First, empirical evidence on the importance of monetary misperceptions deserves to be reexamined. Given the wealth of forecasting data from professional forecasters and financial insitutions which exists today that did not exist 40 years ago when the Lucas islands model was first developed, it should be possible to get a more precise read on the economic impact of surprise money supply shocks.

Second, to match real world dynamics, this model will need to be extended to incorporate persistence of incomplete information. In our framework, incomplete information is reversed after one period. Thus, this model cannot capture the empirical result that the dynamics of inflation and output are smooth over time.

Finally, additional frictions could be incorporated into the model, such as sticky prices or wages, so that anticipated as well as unanticipated monetary shocks affect output.

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Appendix A: Log linearized CIA model

Steady state

To linearize around the steady state, we must first solve for the steady state. From Euler equation (8), we get the steady state real rate of interest is equal to the rate of time preference: $\frac{1+i^{ss}}{1+\pi^{ss}} = 1/\beta$. From market clearing on aggregate resources, we get: $c^{ss} = y^{ss}$.

With a positive nominal interest rate, the CIA constraint (2) binds, and $c^{ss} = \tau^{ss}+m^{ss}/(1+\pi^{ss})$; but in a steady state with constant m, then $\tau^{ss}+m^{ss}/(1+\pi^{ss}) = m^{ss}$. So, we get that $c^{ss} = m^{ss}$.

Combining (5) and (7), $\lambda^{ss} = \frac{\beta}{1+\pi^{ss}} c^{ss-\sigma} = [1-\Phi]\, c^{ss-\sigma}$, where, as defined in the text, $\Phi \equiv 1 - \frac{\beta}{1+\pi^{ss}} = \frac{i^{ss}}{1+i^{ss}}$ is a measure of the steady state deviation from Friedman rule, which equals zero when the Friedman rule is implemented.

This in turn implies using (7) that $\mu^{ss} = \Phi c^{ss-\sigma}$.

From production function (4), $y^{ss} = a^{ss} n^{ss}$. From the labor market clearing condition, using the steady state result for λ and output, $n^{ss} = \left[\frac{1}{\chi}\frac{\beta}{1+\pi^{ss}}\right]^{\frac{1}{\sigma+\eta}} a^{ss\,\frac{1-\sigma}{\sigma+\eta}}$

Linearization

Define the percentage deviation of variable x from its steady state value x^{ss} as $\hat{x}_t = \frac{x_t}{x^{ss}} - 1$. We apply this to all variables excepting interest rates and inflation, which are already percentages, where we instead define $\hat{i}_t = i_t - i^{ss}, \hat{\pi}_t = \pi_t - \pi^{ss}$.

We will use four Uhlig toolkit rules extensively:

1. Product terms don't matter: $uw = u^{ss}(1+\hat{u})w^{ss}(1+\hat{w}) \approx u^{ss}w^{ss}(1+\hat{u}+\hat{w})$

2. Applying repeatedly the above, $u^a = u^{ssa}(1+\hat{u})^a = u^{ssa}(1+a\hat{u})$

3. And, $\log u = \log[u^{ss}(1+\hat{u})] = \log u^{ss} + \log(1+\hat{u}) \approx \log u^{ss} + \hat{u}$

4. In the case of interest rates and inflation, $\frac{1+x_t}{1+x^{ss}} \approx 1 + \hat{x}_t$

First, linearizing the production function gives (9):

$y_t = a_t n_t y^{ss}(1+\hat{y}_t) + a^{ss}n^{ss}(1+\hat{a}_t)(1+\hat{n}_t)$

$\implies 1+\hat{y}_t = (1+\hat{a}_t)(1+\hat{n}_t)$

$$\implies \hat{y}_t = \hat{a}_t + \hat{n}_t \tag{A1}$$

Next, immediately by market clearing, we have $\hat{c}_t = \hat{y}_t$. Further, in an equilibrium with a positive nominal interest rate, the CIA constraint binds and $c_t = \frac{m_{t-1}}{1+\pi_t} + \tau_t = \frac{m_{t-1}}{1+\pi_t} + \frac{M_t - M_{t-1}}{p_t} = \frac{M_{t-1}}{p_{t-1}}\frac{p_{t-1}}{p_t} + \frac{M_t}{p_t} - \frac{M_{t-1}}{p_t} = m_t$ Thus,

$$\hat{c}_t = \hat{m}_t \tag{A2}$$

Combining (5), (7), $c = y$, and linearizing:

$\lambda_t = \beta E_t[y_{t+1}^{-\sigma}/(1+\pi_{t+1})]$

$\implies \lambda^{ss}(1+\hat{\lambda}_t) = \beta \frac{y^{ss-\sigma}}{1+\pi^{ss}} E_t\left[(1-\sigma\hat{y}_{t+1})\frac{1+\pi^{ss}}{1+\pi_{t+1}}\right]$

$\implies 1+\hat{\lambda}_t = E_t\left[\frac{1-\sigma\hat{y}_{t+1}}{1+\hat{\pi}_{t+1}}\right]$

$$\implies \hat{\lambda}_{t+1} = E_t[-\sigma\hat{y}_{t+1} - \hat{\pi}_{t+1}] \tag{A3}$$

From linearizing the labor market equilibrium condition:

$\chi n_t^\eta = \frac{y_t}{n_t}\lambda_t$

$\implies \chi n^{ss\eta}(1+\eta\hat{n}_t) = \frac{y^{ss}}{n^{ss}}\lambda^{ss}(1+\hat{y}_t - \hat{n}_t + \hat{\lambda}_t)$

$$\implies (1+\eta)\hat{n}_t = \hat{y}_t + \hat{\lambda}_t \tag{A4}$$

From linearizing the Euler equation:

$\lambda_t = \beta E_t\left[\lambda_{t+1}\frac{1+i_t}{1+\pi_{t+1}}\right]$

$\implies \lambda^{ss}(1+\hat{\lambda}_t = \beta\lambda^{ss}\frac{1+i^{ss}}{1+\pi^{ss}}E_t\left[(1+\hat{\lambda}_{t+1})\frac{1+i_t}{1+\pi_{t+1}}\frac{1+\pi^{ss}}{1+i^{ss}}\right]$

$\Longrightarrow 1+\hat{\lambda}_t = E_t\left[(1+\hat{\lambda}_{t+1})\frac{1+\hat{i}_t}{1+\hat{\pi}_{t+1}}\right]$

$$\Longrightarrow \hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \hat{i}_t - E_t\hat{\pi}_{t+1} \tag{A5}$$

Combining (7) and (8):
$E_t\left[\frac{\lambda_{t+1}+\mu_{t+1}}{1+\pi_{t+1}}\right] = E_t\left[\lambda_{t+1}\frac{1+i_t}{1+\pi_{t+1}}\right]$
$E_t\left[\frac{\mu_{t+1}-\lambda_{t+1}i_t}{1+\pi_{t+1}}\right] = 0$
$E_t[1+\hat{\mu}_{t+1}-\hat{\pi}_{t+1}] - \frac{\lambda^{ss}}{\mu^{ss}}i^{ss}E_t[1+\hat{\lambda}_{t+1}-\hat{\pi}_{t+1}] - \frac{\lambda^{ss}}{\mu^{ss}}E_t\left[\frac{\hat{i}_t}{1+\hat{\pi}_{t+1}}\right] = 0$
Note $\frac{\lambda^{ss}}{\mu^{ss}} = \frac{1-\Phi}{\Phi} = \frac{1}{i^{ss}}$, so

$$E_t[\hat{\mu}_{t+1} - \hat{\lambda}_{t+1}] = \frac{1-\Phi}{\Phi}E_t\hat{i}_t \tag{A6}$$

(A1)-(A6) correspond to (9)-(14).

Appendix B: Unconstrained full-information output

Full-information, unconstrained output is defined as the level of output which would occur in the absence of any information frictions and in the absence of the CIA constraint. This can be derived using the labor market equilibrium condition, (6) (rather than (18), which includes information frictions). The condition is:
$\chi {n_t^i}^\eta = \lambda_t^i \frac{y_t^i}{n_t^i}$
Substitute out λ using the marginal utility of wealth equation (5) and market clearing $c = y$:
$\frac{\chi {n_t^i}^\eta}{{y_t^i}^{-\sigma} - \mu_t^i} = \frac{y_t^i}{n_t^i}$
As mentioned, we want unconstrained output, so we turn off the CIA constraint by setting $\mu_t^i = \mu^{ss} = 0$:
$\chi {n_t^i}^\eta {y_t^i}^\sigma = \frac{y_t^i}{n_t^i}$

Next, substitute out for labor using the production function $n_t^i = y_t^i / a_t \aleph_t^i$. This gives:

$\chi \left(\frac{y_t^i}{a_t \aleph_t^i} \right)^\eta y_t^{i\sigma} = a_t \aleph_t^i$

Rearranging, and denoting this level of output as the full-information unconstrained level y_t^{if}, we have

$y_t^{if} = \left[\frac{(a_t \aleph_t^i)^{1+\eta}}{\chi} \right]^{\frac{1}{\eta+\sigma}}$

Linearizing this gives the full-information unconstrained level of output cited in equation (26):

$$\hat{y}_t^{i,f} = \frac{1+\eta}{\sigma+\eta} \left[\hat{a}_t + \hat{\aleph}_t^i \right]$$

The derivation for aggregate full-information unconstrained output is exactly analogous, simply ignoring the superscript i's and $\aleph^i$.

Appendix C: The welfare approximation

To take a second order approximation, we begin by defining the following notation:

X^{ss} : steady state

$\tilde{X}_t = X_t - X^{ss}$

$\hat{X}_t = \log X_t - \log X^{ss}$

Given this notation,

$$\frac{X_t}{X^{ss}} \approx 1 + \log \frac{X_t}{X^{ss}} + \frac{1}{2} \left[\log \frac{X_t}{X^{ss}} \right]^2 = 1 + \hat{X}_t + \frac{1}{2} \hat{X}_t^2$$

Furthermore, since we can write $\tilde{X}_t = X^{ss} \left(\frac{X_t}{X^{ss}} - 1 \right)$, we get that

$$\tilde{X}_t \approx X^{ss} \left(\hat{X}_t + \frac{1}{2} \hat{X}_t^2 \right)$$

For the sake of clarity, we drop i superscripts below.

We wish to approximate period utility around its steady state, where period utility is

$$U_t = U(c_t, n_t) = \frac{1}{1-\sigma}c_t^{1-\sigma} - \frac{\chi}{1+\eta}n_t^{1+\eta} \equiv W(c_t) - V(n_t)$$

We first do a second order approximation of $W(c_t)$. Taking a Taylor approximation,

$W(c_t) \approx W(c^{ss}) + W_c(c^{ss})\tilde{c}_t + \frac{1}{2}W_{cc}(c^{ss})^2\tilde{c}_t^2$

Using the notation and results above to substitute,

$W(c_t) \approx W(c^{ss}) + W_c(c^{ss})\left(\hat{c}_t + \frac{1}{2}\hat{c}_t^2\right) + \frac{1}{2}W_{cc}(c^{ss})^2\left(\hat{c}_t + \frac{1}{2}\hat{c}_t^2\right)^2$

Dropping terms of third order or higher,

$W(c_t) \approx W(c^{ss}) + W_c(c^{ss})\left(\hat{c}_t + \frac{1}{2}\hat{c}_t^2\right) + \frac{1}{2}W_{cc}(c^{ss})^2\hat{c}_t^2$

Given our choice of utility function, $W_{cc} = \sigma W_c/c$, so

$$W(c_t) \approx W(c^{ss}) + W_c(c^{ss})c^{ss}\left[\hat{c}_t + \frac{1}{2}(1-\sigma)\hat{c}_t^2\right]$$

In parallel, one can derive that

$$V(n_t) \approx V(n^{ss}) + V_n(n^{ss})n^{ss}\left[\hat{n}_t + \frac{1}{2}(1+\eta)\hat{n}_t^2\right]$$

Combining these two expressions to get period welfare, and noting that $c^{ss} = y^{ss} = n^{ss}; \hat{c}_t = \hat{y}_t; \hat{n}_t = \hat{y}_t - \hat{a}_t$, and that $\frac{V_n(n^{ss})}{U_c(c^{ss})} = \frac{y^{ss}}{n^{ss}} = 1$, we get that

$$\begin{aligned}
U_t - U^{ss} &= W_c(y^{ss})y^{ss}\left[\hat{y}_t + \frac{1}{2}(1-\sigma)\hat{y}_t^2\right] - V_n(n^{ss})y^{ss}\left[\hat{y}_t - \hat{a}_t + \frac{1}{2}(1+\eta)(\hat{y}_t - \hat{a}_t)^2\right] \\
&= U_c(y^{ss})y^{ss}\left\{\frac{1}{2}(-\sigma-\eta)\hat{y}_t^2 + \hat{a}_t - \frac{1}{2}(1+\eta)(\hat{y}_t^2 - 2\hat{y}_t\hat{a}_t + \hat{a}_t^2)\right\} \\
&= U_c(y^{ss})y^{ss}\left\{[1+(1+\eta)\hat{y}_t]\hat{a}_t - \frac{1}{2}[\sigma+\eta]\hat{y}_t^2 - \frac{1}{2}(1+\eta)\hat{a}_t^2\right\}
\end{aligned}$$

We now manipulate this equation, adding and subtracting terms independent of policy, to achieve an equation of the desired form:

$2\frac{1}{1+\eta}\frac{U_t - U^{ss}}{U_c(y^{ss})y^{ss}} = 2\frac{a_t}{1+\eta} + 2\hat{y}_t\hat{a}_t - \frac{\sigma+\eta}{1+\eta}\hat{y}_t^2 - \hat{a}_t^2$

$2\frac{1+\eta}{\sigma+\eta}\frac{1}{1+\eta}\frac{U_t - U^{ss}}{U_c(y^{ss})y^{ss}} = 2\frac{1+\eta}{\sigma+\eta}\frac{a_t}{1+\eta} + 2\frac{1+\eta}{\sigma+\eta}\hat{y}_t\hat{a}_t - \hat{y}_t^2 - \frac{1+\eta}{\sigma+\eta}\hat{a}_t^2$

Or,

$$
\begin{aligned}
\xi\left[U_t - U^{ss}\right] &= -\frac{1}{2}\left[\hat{y}_t - \hat{y}_t^f\right]^2 + t.i.p. \\
t.i.p &= \frac{1+\eta}{\sigma+\eta}\frac{\hat{a}_t}{1+\eta} - \frac{1}{2}\left(\frac{1+\eta}{\sigma+\eta}\right)\hat{a}_t^2 - \frac{1}{2}\left(\frac{1+\eta}{\sigma+\eta}\right)^2\hat{a}_t^2 \\
\xi &= \frac{1}{\sigma+\eta}\frac{1}{U_c(y^{ss})y^{ss}}
\end{aligned}
$$

This is the equation shown in (28).

Appendix D: The recursive laws of motion

The relevant system of equations is:

$$\hat{y}_t^i = \hat{n}_t^i + \hat{a}_t + \hat{\aleph}_t^i \tag{a}$$

$$\hat{\lambda}_t^i = E_t^i[-\sigma\hat{y}_{t+1} - \hat{\pi}_{t+1}] \tag{b}$$

$$(1+\eta)\hat{n}_t^i = \hat{y}_t^i + \hat{\lambda}_t^i \tag{c}$$

$$\hat{y}_t^i = \hat{M}_t^i - \hat{p}_t^i \tag{d}$$

From (c) and (a), we get that
$\hat{\lambda}_t^i = (1+\eta)\hat{n}_t^i - \hat{y}_t^i = (1+\eta)\hat{n}_t^i - \hat{y}_t^i = \eta\hat{y}_t^i - (1+\eta)[\hat{a}_t + \hat{\aleph}_t^i]$.
Substituting this into (b), we get
$\eta\hat{y}_t^i - (1+\eta)[\hat{a}_t + \hat{\aleph}_t^i] = E_t^i[-\sigma\hat{y}_{t+1} - \hat{\pi}_{t+1}]$
Substituting out for output using the CIA constraint, and for inflation in terms of the price level,

$$\eta[\hat{M}_{t-1} - \hat{p}_{t-1}] - (1+\eta)[\hat{a}_t + \hat{\aleph}_t^i] = E_t^i\left[-\sigma\left(\hat{M}_{t+1} - \hat{p}_{t+1}\right) - \hat{p}_{t+1} + \hat{p}_t^i + \hat{T}_{t+1}\right] \tag{*}$$

Note the expression for conditional expected money supply at t and $t+1$:

$$
\begin{aligned}
E_t^i \hat{M}_t &= \rho_m \hat{M}_{t-1} + v_t + E_t^i u_t + \phi E_t^i \hat{a}_t \\
&= \rho_m \hat{M}_{t-1} + v_t + \kappa(u_t + u_t^i) + \phi\omega(\hat{a}_t + \hat{\aleph}_t^i) \\
E_t^i \hat{M}_{t+1} &= \rho_m E_t^i \hat{M}_t + \phi E_t^i \hat{a}_{t+1} \\
&= \rho_m \left[\hat{M}_{t-1} + v_t + \kappa(u_t + u_t^i)\right] + \phi\omega(\rho_m + \rho_a)(\hat{a}_t + \hat{\aleph}_t^i)
\end{aligned}
$$

Substituting this into (*) and consolidating terms,

$$
\begin{aligned}
(1+\eta)\hat{p}_t^i = \eta \hat{M}_t^i \\
&+ \left[-(1+\eta) - (1-\sigma)\phi\omega(\rho_m + \rho_a) + \phi\omega\right](\hat{a}_t + \hat{\aleph}_t^i) \\
&+ \left[-(1-\sigma)\rho_m + 1\right]\left[\rho_m \hat{M}_{t-1} + v_t + \kappa(u_t + u_t^i)\right] \\
&- (\sigma - 1) E_t^i \hat{p}_{t+1}
\end{aligned}
$$

Substituting in the island-specific money supply rule, $\hat{M}_t^i = \rho_m \hat{M}_{t-1} + v_t + u_t + u_t^i + \phi \hat{a}_t + \phi^i \hat{\aleph}_t^i$, this becomes

$$
\begin{aligned}
(1+\eta)\hat{p}_t^i = \eta \hat{M}_t^i \\
&+ \left[-(1+\eta) - (1-\sigma)\phi\omega(\rho_m + \rho_a) + \phi\omega(1 + \eta/\omega)\right]\hat{a}_t \\
&+ \left[-(1+\eta) - (1-\sigma)\phi\omega(\rho_m + \rho_a) + \phi\omega + \eta\phi^i\right]\hat{\aleph}_t^i \\
&+ \left[-(1-\sigma)\rho_m + 1 + \eta\right]\rho_m \hat{M}_{t-1} \\
&+ \left[-(1-\sigma)\rho_m + 1 + \eta\right] v_t \\
&+ \left[-(1-\sigma)\rho_m + 1 + \frac{\eta}{\kappa}\right]\kappa(u_t + u_t^i) \\
&- (\sigma - 1) E_t^i \hat{p}_{t+1} \qquad (**)
\end{aligned}
$$

We now solve via method of undetermined coefficients: suppose

$$
\hat{p}_t^i = a_1 \hat{M}_{t-1} + a_2 v_t + a_3 u_t + a_4 u_t^i + a_5 \hat{a}_t + a_6 \hat{\aleph}_t^i
$$

Note that

$\hat{p}_t = a_1\hat{M}_{t-1} + a_2 v_t + a_3 u_t + a_5 \hat{a}_t$

So that

$$E_t^i \hat{p}_{t+1} = a_1[\rho_m \hat{M}_{t-1} + v_t + \kappa(u_t + u_t^i)] + [a_1\phi + a_5\rho_a]\omega[\hat{a}_t + \hat{\aleph}_t^i]$$

We now isolate terms and solve for coefficients $a_1, ..., a_6$ by substituting the expressions for $\hat{p}_t$ and $E_t^i \hat{p}_{t+1}$ into (**).

For $\hat{M}_{t-1}$,

$(1+\eta)a_1 = [-(1-\sigma)\rho_m + 1 + \eta]\rho_m - (\sigma - 1)a_1\rho_m$

$\implies a_1 = \rho_m$

For v_t,

$(1+\eta)a_2 = [-(1-\sigma)\rho_m + 1 + \eta] - (\sigma - 1)a_1$

$\implies a_2 = 1$

For u_t,

$(1+\eta)a_3 = [-(1-\sigma)\rho_m + 1 + \frac{\eta}{\kappa}]\kappa - (\sigma - 1)\rho_m\kappa$

$\implies a_3 = \frac{\eta+\kappa}{\eta+1}$

For u_t^i,

$(1+\eta)a_4 = [-(1-\sigma)\rho_m + 1 + \frac{\eta}{\kappa}]\kappa - (\sigma - 1)\rho_m\kappa$

$\implies a_4 = \frac{\eta+\kappa}{\eta+1}$

For $\hat{a}_t$,

$(1+\eta)a_5 = [-(1+\eta) - (1-\sigma)\phi\omega(\rho_m + \rho_a) + (1+\eta/\omega)\phi\omega] - (\sigma - 1)\omega[a_1\phi + a_5\rho_a]$

$\implies a_5 = -\frac{\eta+1}{\eta+1+(\sigma-1)\omega\rho_a} + \phi\frac{\eta+\omega+(\sigma-1)\omega\rho_a}{\eta+1+(\sigma-1)\omega\rho_a}$

For $\hat{\aleph}_t^i$,

$(1+\eta)a_6 = [-(1+\eta) - (1-\sigma)\phi\omega(\rho_m + \rho_a) + \phi\omega + \eta\phi^i] - (\sigma - 1)\omega[a_1\phi + a_5\rho_a]$

$\implies a_6 = -1 + \frac{\eta}{\eta+1}\phi^i + \phi\frac{\omega+(\sigma-1)\omega\rho_a}{\eta+1} - \frac{\sigma-1}{\eta+1}\omega\rho_a a_5$

These are the results shown in equation (29)-(32).